PG – 816

II Semester M.Sc. Degree Examination, June 2015
(RNS) (2011-12 and Onwards)
MATHEMATICS
M – 201 : Algebra – II

Time : 3 Hours Max. Marks : 80

Instructions : *i) Answer* ***any five(5) full*** *questions choosing at least* ***two*** *from* ***each*** *Part.*
ii) ***All*** *questions carry* ***equal*** *marks.*

PART – A

1. a) Let K be an extension of a field F and $a \in K$ be algebraic over F and of degree n. Prove that $[F(a):F] = n$. **6**

 b) Prove that every finite extension K of a field F is algebraic and may be obtained from F by the adjunction of finitely many algebraic elements. **6**

 c) Prove that $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$. **4**

2. a) Let p(x) be a polynomial in F[x] of degree $n \geq 1$ and is irreducible over F, then prove that there is an extension E of F such that $[E : F] = n$, in which p(x) has a root. **6**

 b) Define splitting field of a polynomial over a field F. Determine the splitting field of $x^2 + \alpha x + \beta$ over the field Q and hence verify $s(x^2 + 2 : Q) = Q(\sqrt{2}\ i)$. **5**

 c) If ‘p’ is a prime number, prove that the splitting field over F, the field of rationals, of the polynomial $x^p - 1$ is of degree $p - 1$. **5**

3. a) Prove that a regular pentagon is constructible by using straight edge and compass. **6**

 b) Prove that a polynomial $f(x) \in F[x]$ has a multiple root if and only if f(x) and $f'(x)$ have a non-trivial common factor. **5**

 c) If F is a field of characteristic zero and a, b are algebraic over F, then prove that F(a, b) is a simple extension of F. **5**

4. State and prove the fundamental theorem of Galois theory. **16**

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PART – B

5. a) If A is an algebra with unit element over F, then show that A is isomorphic to a subalgebra of A(V) for some vector space V over F. **6**

   b) Give an example to show that an element in A(V) is right invertible but not invertible. **4**

   c) If V is a finite dimensional vector space over F and T, S $\in$ A(V) with S regular, then prove that both T and STS^{-1} have the same minimal polynomial over F. **6**

6. a) If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ in F are distinct characteristic roots of T $\in A_F(V)$ and if $V_1, V_2, \ldots, V_k$ are characteristic vectors of T belonging to $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ respectively, then prove that $V_1, V_2, \ldots, V_k$ are linearly independent. **4**

   b) If $\lambda \in$ F is a characteristic root of T $\in$ A(V), then for any q(x) $\in$ F(x), prove that $q(\lambda)$ is a characteristic root of q(T). **4**

   c) Let F be a field and let V be the set of all polynomials in X of degree 3 or less over F. On V let D be defined by $(\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3)D = \beta_1 + 2\beta_2 x + 3\beta_3 x^2$. Let $M_1(D)$ be the matrix in the basis $\{1, x, x^2, x^3\}$ and let $M_2(D)$ be the matrix in the basis $\{1, 1+x, 1+x^2, 1+x^3\}$. Find a matrix G such that $M_2(D) = Cm_1(D)C^{-1}$. **8**

7. a) If T $\in$ A(V) has all its characteristic roots in F, then prove that there is a basis of V in which the matrix T is triangular. **8**

   b) Define a nilpotent transformation. If T $\in$ A(V) is nilpotent, of index n_1, then prove that there exists a basis of V such that the matrix of T in this basis has the form.

$$\begin{bmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & M_{n_r} \end{bmatrix},$$

   where $n_1 \geq n_2 \geq, \ldots \geq n_r$ and $n_1 + n_2 + \ldots + n_r = \dim_F V$. **8**

8. a) Define a unitary transformation T. Prove that linear transformation, T on V is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V. **6**

   b) State and prove Sylvester's law of inertia. **10**